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GENERALIZED INVERSION OF MODIFIED MATRICES*

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Abstract. For an $m \times n$ complex matrix A and two columns, c and d, representations for the Moore–Penrose inverse of the matrix $A + cd^*$ are given for all possible cases. Moreover, each representation involves only A, $A^\dagger$, c, d, and their conjugate transposes.

1. Introduction. For a square, nonsingular matrix A, it is well known [2, pp. 173–178] that when A is modified by a matrix of rank 1 to produce $M = A + cd^*$, the inverse of M, if it exists, is given by the formula

$$M^{-1} = A^{-1} - \frac{1}{\beta}A^{-1}cd^*A^{-1}, \tag{1.1}$$

where $\beta = 1 + d^*A^{-1}c$. By means of (1.1), one can alter one or more of the elements of A and still use A^{-1} to invert the modified matrix [1], [3, p. 79], [5]. Also, (1.1) is the basis for the inversion schemes known as rank annihilation or the reinforcement method [6], [2, pp. 173–178]. However, if one is dealing with either a rectangular matrix or a square singular matrix A, as is often the case, and he has previously obtained the Moore–Penrose inverse [4] $A^\dagger$ of A, it is not always possible to obtain the Moore–Penrose inverse of the modified matrix $A + cd^*$ by using (1.1) with (-1) replaced by $(\dagger)$. Earlier work by Cline [8], allowed one to obtain the Moore–Penrose inverse of $A + cd^*$ in the special cases when $Ad^*c = 0$ or else when $c = d$ and $A = SS^*$ for some S. However, the general expression for $(A + cd^*)^\dagger$ has not yet been given.

In this paper, we present expressions for $(A + cd^*)^\dagger$ which cover all possible cases. Moreover, each of our expressions for $(A + cd^*)^\dagger$ are of the form

$$(A + cd^*)^\dagger = A^\dagger + G,$$

where G is a matrix obtained from only sums and products of A, $A^\dagger$, c, d, and their conjugate transposes, so that our expressions may be used to obtain the Moore–Penrose inverse of a modified matrix in much the same way (1.1) is used in the nonsingular case.

2. Notation. For a given $m \times n$ complex matrix A and two columns c and d, we adopt the following notation:

$\overline{(\cdot)}$ —the complex conjugate,
$*$ —the conjugate transpose,
$\dagger$ —the Moore–Penrose inverse,
$\|\cdot\|$ —the Euclidean norm,
$R(\cdot)$—the range or column space,
k —the column $A^\dagger c$,
h —the row $d^*A^\dagger$,
u —the column $(I - AA^\dagger)c$,
v —the row $d^*(I - A^\dagger A)$,
β —the scalar $1 + d^*A^\dagger c$.

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Throughout, we make use of the fact that for a nonzero vector x, its Moore–Penrose inverse is given by

$$x^{\dagger} = \frac{x^*}{\|x\|^2}.$$

3. Main results. In order to investigate the structure of the matrix $(A + cd^*)^{\dagger}$, there are six distinct cases to consider and we enumerate them as follows:

(i) $c \notin R(A)$ and $d \notin R(A^*)$;
(ii) $c \in R(A)$ and $d \notin R(A^*)$ and $\beta = 0$;
(iii) $c \in R(A)$ and d arbitrary and $\beta \neq 0$;
(iv) $c \notin R(A)$ and $d \in R(A^*)$ and $\beta = 0$;
(v) c arbitrary and $d \in R(A^*)$ and $\beta \neq 0$;
(vi) $c \in R(A)$ and $d \in R(A^*)$ and $\beta = 0$.

THEOREM 1.

$$(A + cd^*)^{\dagger} = A^{\dagger} - ku^{\dagger} - v^{\dagger}h + \beta v^{\dagger}u^{\dagger}, \tag{3.1}$$

when $c \notin R(A)$ and $d \notin R(A^*)$.

THEOREM 2.

$$(A + cd^*)^{\dagger} = A^{\dagger} - kk^{\dagger}A^{\dagger} - v^{\dagger}h, \tag{3.2}$$

when $c \in R(A)$, $d \notin R(A^*)$ and $\beta = 0$.

THEOREM 3. *Let*

$$p_1 = -\frac{\|k\|^2}{\bar{\beta}}v^* - k, \qquad q_1^* = -\frac{\|v\|^2}{\bar{\beta}}k^*A^{\dagger} - h,$$

and

$$\sigma_1 = \|k\|^2\|v\|^2 + |\beta|^2.$$

Then,

$$(A + cd^*)^{\dagger} = A^{\dagger} + \frac{1}{\bar{\beta}}v^*k^*A^{\dagger} - \frac{\bar{\beta}}{\sigma_1}p_1q_1^*, \tag{3.3}$$

when $c \in R(A)$ and $\beta \neq 0$.

THEOREM 4.

$$(A + cd^*)^{\dagger} = A^{\dagger} - A^{\dagger}h^{\dagger}h - ku^{\dagger}, \tag{3.4}$$

when $c \notin R(A)$, $d \in R(A^*)$ and $\beta = 0$.

THEOREM 5. *Let*

$$p_2 = -\frac{\|u\|^2}{\bar{\beta}}A^{\dagger}h^* - k, \qquad q_2^* = -\frac{\|h\|^2}{\bar{\beta}}u^* - h,$$

and

$$\sigma_2 = \|h\|^2\|u\|^2 + |\beta|^2.$$

Then,

$$(A + cd^*)^{\dagger} = A^{\dagger} + \frac{1}{\bar{\beta}}A^{\dagger}h^*u^* - \frac{\bar{\beta}}{\sigma_2}p_2q_2^*, \tag{3.5}$$

when $d \in R(A^*)$ and $\beta \neq 0$.

THEOREM 6.

$$(A + cd^*)^\dagger = A^\dagger - kk^\dagger A^\dagger - A^\dagger h^\dagger h + (k^\dagger A^\dagger h^\dagger)kh, \tag{3.6}$$

when $c \in R(A)$, $d \in R(A^*)$ *and* $\beta = 0$.

Before proving the theorems, two preliminary facts are needed, and we state these as lemmas.

LEMMA 1.

$$\operatorname{rank}(A + cd^*) = \operatorname{rank}\begin{bmatrix} A & u \\ v & -\beta \end{bmatrix} - 1.$$

Proof. This follows immediately from the factorization

$$\begin{bmatrix} A + cd^* & c \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ h & 1 \end{bmatrix}\begin{bmatrix} A & u \\ v & -\beta \end{bmatrix}\begin{bmatrix} I & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ d^* & 1 \end{bmatrix}.$$

It is worth noting that $c \in R(A)$ if and only if $u = 0$ and $d \in R(A^*)$ if and only if $v = 0$.

LEMMA 2. *If M and X are matrices such that $XMM^\dagger = X$ and $M^\dagger M = XM$, then $X = M^\dagger$.*

Proof.

$$M^\dagger = (M^\dagger M)M^\dagger = XMM^\dagger = X.$$

We now proceed with the proof of the theorems. Throughout, we assume $c \neq 0$ and $d \neq 0$.

Proof of Theorem 1. Let X_1 denote the right-hand side of (3.1) and let $M = A + cd^*$. The proof is showing that X_1 satisfies the four Penrose conditions: (1) $MX_1M = M$, (2) $X_1MX_1 = X_1$, (3) $(MX_1)^* = MX_1$, and (4) $(X_1M)^* = X_1M$. Using $Av^\dagger = 0$, $d^*v^\dagger = 1$, $d^*k = \beta - 1$, and $c - Ak = u$, it is easy to see that

$$MX_1 = AA^\dagger + uu^\dagger$$

so that (3) holds. Using $u^\dagger A = 0$, $u^\dagger c = 1$, $hc = \beta - 1$, and $d^* - hA = v$, one obtains

$$X_1M = A^\dagger A + v^\dagger v$$

and hence (4) holds. Conditions (1) and (2) are now easily verified.

Proof of Theorem 2. Let X_2 denote the right-hand side of (3.2). By using $Ak = c$, $Av^\dagger = 0$, $d^*v^\dagger = 1$, and $d^*k = -1$ it is seen that

$$(A + cd^*)X_2 = AA^\dagger,$$

which is Hermitian. From the facts that $k^\dagger A^\dagger A = k^\dagger$, $hc = -1$, and $d^* - hA = v$, it follows that

$$X_2(A + cd^*) = A^\dagger A - kk^\dagger + v^\dagger v,$$

which is also Hermitian. The first and second Penrose conditions are now easily verified.

Proof of Theorem 3. This case is the most difficult. In this case, $c \in R(A)$ and hence it follows that $R(A + cd^*) \subset R(A)$ and $u = 0$. Since $\beta \neq 0$, it is clear from

Lemma 1 that

$$\operatorname{rank}(A + cd^*) = \operatorname{rank}(A)$$

so that $R(A + cd^*) = R(A)$ and therefore

$$(A + cd^*)(A + cd^*)^\dagger = AA^\dagger \tag{3.7}$$

because $AA^\dagger$ is the unique orthogonal projector onto $R(A)$. (For a discussion of projectors, see [7, p. 106].) Let X_3 denote the right-hand side of (3.3). Because $q_1^* AA^\dagger = q_1^*$, it is immediate from (3.7) that

$$X_3(A + cd^*)(A + cd^*)^\dagger = X_3$$

and hence the first condition of Lemma 2 is satisfied.

To show that the second condition of Lemma 2 is also satisfied, we first show that

$$(A + cd^*)^\dagger(A + cd^*) = A^\dagger A - kk^\dagger + p_1 p_1^\dagger .$$

The matrix $A^\dagger A - kk^\dagger + p_1 p_1^\dagger$ is Hermitian and idempotent. The fact that it is Hermitian is clear and the fact that it is idempotent follows by direct computation using $A^\dagger Ak = k$, $A^\dagger A p_1 = -k$, and $kk^\dagger p_1 = -k$. Since the rank of an idempotent matrix is equal to its trace ([9, p. 224]) and since trace is a linear function, it follows that

$$\begin{aligned}\operatorname{rank}(A^\dagger A - kk^\dagger + p_1 p_1^\dagger) &= \operatorname{trace}(A^\dagger A - kk^\dagger + p_1 p_1^\dagger) \\ &= \operatorname{trace}(A^\dagger A) - \operatorname{trace}(kk^\dagger) + \operatorname{trace}(p_1 p_1^\dagger).\end{aligned}$$

Now, $kk^\dagger$ and $p_1 p_1^\dagger$ are idempotent matrices of rank = trace = 1 and $A^\dagger A$ is an idempotent matrix whose rank is equal to rank (A), so that

$$\operatorname{rank}(A^\dagger A - kk^\dagger + p_1 p_1^\dagger) = \operatorname{rank}(A + cd^*). \tag{3.8}$$

Using the facts $Ak = c$, $Ap_1 = -c$, $d^*k = \beta - 1$, $d^*p_1 = 1 - \sigma_1 \bar{\beta}^{-1}$, and $d^*A^\dagger A = d^* - v$, one obtains

$$(A + cd^*)(A^\dagger A - kk^\dagger + p_1 p_1^\dagger) = A + cd^* - c(v + \beta k^\dagger + \sigma_1 \bar{\beta}^{-1} p_1^\dagger).$$

Now, $\|p_1\|^2 = \|k\|^2 \sigma_1 |\beta|^{-2}$, so that $\sigma_1 \bar{\beta}^{-1} \|p_1\|^{-2} = \beta \|k\|^{-2}$ and hence

$$\sigma_1 \bar{\beta}^{-1} p_1^\dagger + \beta \|k\|^{-2} p_1^* = -v - \beta k^\dagger .$$

Thus,

$$(A + cd^*)(A^\dagger A - kk^\dagger + p_1 p_1^\dagger) = A + cd^*.$$

Because $A^\dagger A - kk^\dagger + p_1 p_1^\dagger$ is an orthogonal projector, it follows that

$$R(A^* + dc^*) \subset R(A^\dagger A - kk^\dagger + p_1 p_1^\dagger).$$

By virtue of (3.8), we conclude that

$$R(A^* + dc^*) = R(A^\dagger A - kk^\dagger + p_1 p_1^\dagger),$$

and hence, $(A^* + dc^*)(A^* + dc^*)^\dagger = A^\dagger A - kk^\dagger + p_1 p_1^\dagger$, or equivalently,

$$(A + cd^*)^\dagger(A + cd^*) = A^\dagger A - kk^\dagger + p_1 p_1^\dagger .$$

To show that $X_3(A + cd^*) = A^\dagger A + p_1 p_1^\dagger - kk^\dagger$, we compute $X_3(A + cd^*)$ after observing that $k^* A^\dagger A = k^*, q_1^* c = 1 - \sigma_1 \bar{\beta}^{-1}$, and $q_1^* A + d^* = -\|v\|^2 \bar{\beta}^{-1} k^* + v$. Now,

$$\begin{aligned}
X_3(A + cd^*) &= A^\dagger A + \frac{1}{\bar{\beta}} v^* k^* - \frac{\bar{\beta}}{\sigma_1} p_1 q_1^* A + \left(k + \frac{\|k\|^2}{\bar{\beta}} v^*\right) d^* - \frac{\bar{\beta}}{\sigma_1} p_1 q_1^* c d^* \\
&= A^\dagger A + \frac{1}{\bar{\beta}} v^* k^* - \frac{\bar{\beta}}{\sigma_1} p_1 q_1^* A - p_1 d^* - \frac{\bar{\beta}}{\sigma_1} p_1 d^* + p_1 d^* \\
&= A^\dagger A + \frac{1}{\bar{\beta}} v^* k^* - \frac{\bar{\beta}}{\sigma_1} p_1 (q_1^* A + d^*) \\
&= A^\dagger A + \frac{1}{\bar{\beta}} v^* k^* - \frac{\bar{\beta}}{\sigma_1} p_1 (v - \|v\|^2 \bar{\beta}^{-1} k^*).
\end{aligned}$$

Write v as $v = -\beta \|k\|^{-2}(p_1^* + k^*)$ and substitute this in the expression in parentheses and use the fact that $\|p_1\|^{-2} = |\beta|^2 \sigma_1^{-1} \|k\|^{-2}$ to obtain

$$X_3(A + cd^*) = A^\dagger A + \frac{1}{\bar{\beta}} v^* k^* + p_1 p_1^\dagger + \frac{1}{\|k\|^2} p_1 k^*.$$

Since

$$\frac{1}{\bar{\beta}} v^* + \frac{1}{\|k\|^2} p_1 = \frac{1}{\bar{\beta}} v^* - \frac{1}{\bar{\beta}} v^* - \frac{1}{\|k\|^2} k = -\frac{1}{\|k\|^2} k,$$

we arrive at

$$X_3(A + cd^*) = A^\dagger A + p_1 p_1^\dagger - kk^\dagger.$$

Thus

$$(A + cd^*)^\dagger (A + cd^*) = X_3(A + cd^*)$$

so that $X_3 = (A + cd^*)^\dagger$, from Lemma 2.

Proof of Theorem 4. This case is the dual of Theorem 2 in the sense that it follows by considering conjugate transposes and using the fact that $M^{\dagger *} = M^{*\dagger}$ for every matrix M. From Theorem 2,

$$(A^* + dc^*)^\dagger = A^{*\dagger} - h^* h^{*\dagger} A^{*\dagger} - u^{*\dagger} k^*$$

and hence,

$$(A + cd^*)^\dagger = (A + cd^*)^{*\dagger *} = (A^{*\dagger} - h^* h^{*\dagger} A^{*\dagger} - u^{*\dagger} k^*)^* = A^\dagger - A^\dagger h^\dagger h - ku^\dagger.$$

Proof of Theorem 5. This case is the dual of Theorem 3 in the sense that it follows from Theorem 3 by considering the conjugate transpose of $A + cd^*$ in a manner similar to that used in the proof of Theorem 4.

Proof of Theorem 6. Each of the matrices

$$AA^\dagger - h^\dagger h \quad \text{and} \quad A^\dagger A - kk^\dagger$$

is an orthogonal projector. The fact that they are idempotent follows from $AA^\dagger h^\dagger = h^\dagger$, $hAA^\dagger = h$, $A^\dagger A k = k$ and $k^\dagger A^\dagger A = k^\dagger$. It is clear that each is

Hermitian. Moreover, the rank of each is equal to its trace and hence each has rank equal to rank $(A) - 1$. Also, since $u = 0$, $v = 0$, and $\beta = 0$, it follows from Lemma 1 that

$$\operatorname{rank}(A + cd^*) = \operatorname{rank}(A) - 1.$$

Hence,

$$\operatorname{rank}(A + cd^*) = \operatorname{rank}(AA^\dagger - h^\dagger h) = \operatorname{rank}(AA^\dagger - k^\dagger k). \tag{3.9}$$

With the facts $AA^\dagger c = c$, $hc = -1$, and $hA = d^*$, it is easy to see that

$$(AA^\dagger - h^\dagger h)(A + cd^*) = (A + cd^*),$$

so that $R(A + cd^*) \subset R(AA^\dagger - h^\dagger h)$. Likewise, using $d^*A^\dagger A = d^*$, $d^*k = -1$, and $Ak = c$, one sees that

$$(A + cd^*)(A^\dagger A - kk^\dagger) = A + cd^*,$$

and hence $R(A^* + dc^*) \subset R(A^\dagger A - kk^\dagger)$. By virtue of (3.9), it now follows that

$$(A + cd^*)(A + cd^*)^\dagger = AA^\dagger - h^\dagger h \tag{3.10}$$

and

$$(A + cd^*)^\dagger(A + cd^*) = A^\dagger A - kk^\dagger. \tag{3.11}$$

If X_6 denotes the right-hand side of (3.6), use (3.10) and the fact that $hAA^\dagger = h$ to obtain

$$X_6(A + cd^*)(A + cd^*)^\dagger = X_6,$$

which is the first condition of Lemma 2. Use $k^\dagger A^\dagger A = k^\dagger$, $hA = d^*$, and $hc = -1$ to obtain

$$X_6(A + cd^*) = A^\dagger A - kk^\dagger.$$

By virtue of (3.11), we have that the second condition of Lemma 2 is satisfied and hence $X_6 = (A + cd^*)^\dagger$.

COROLLARY (The analogue of (1.1)). *When $c \in R(A)$, $d \in R(A^*)$, and $\beta \neq 0$, the Moore–Penrose inverse of $A + cd^*$ is given by*

$$(A + cd^*)^\dagger = A^\dagger - \frac{1}{\beta}A^\dagger cd^*A^\dagger = A^\dagger - \frac{1}{\beta}kh.$$

Proof. This is obtained from Theorem 3 by setting $v = 0$ or from Theorem 5 by setting $u = 0$.

4. Special cases. In many applications, particularly in statistical applications, one deals not so much with the Moore–Penrose inverse, but rather with a non-unique generalized inverse which satisfies only the first Penrose condition: i.e., for a given matrix A, A^- is called a generalized inverse (g-inverse) for A if $AA^-A = A$. It is well known [7, p. 40] that if $1 + d^*A^-c = \beta \neq 0$ for some A^- and either $c \in R(A)$ or $d \in R(A^*)$, then a g-inverse for $A + cd^*$ is given by

$$(A + cd^*)^- = A^- - \beta^{-1}A^-cd^*A^- \qquad \text{(the analogue of (1.1))}.$$

However, expressions for g-inverses of $A + cd^*$ in the other cases have never been given. It is not difficult to examine each of the expressions for $(A + cd^*)^\dagger$ given earlier and decide which terms of each expression can be omitted and which should be kept, so that just the first Penrose condition is satisfied. We state these observations below.

THEOREM 7. *Let A be an $m \times n$ complex matrix and let c and d be $m \times 1$ and $n \times 1$ columns, respectively. Let A^- be some g-inverse for A; let E and F denote matrices*

$$E = I - AA^- \quad \textit{and} \quad F = I - A^-A;$$

and let $\beta = 1 + d^*A^-c$. A g-inverse for $A + cd^*$ is given by the following:

$$(A + cd^*)^- = A^- - \frac{A^-cc^*E}{c^*Ec} - \frac{Fdd^*A^-}{d^*Fd} + \beta\frac{Fdc^*E}{(c^*Ec)(d^*Fd)},$$

$$\textit{when } c \notin R(A), \quad d \notin R(A^*);$$

$$(A + cd^*)^- = A^- - \frac{Fdd^*A^-}{d^*Fd},$$

$$\textit{when } \beta = 0, \quad c \in R(A), \quad d \notin R(A^*);$$

$$(A + cd^*)^- = A^- - \beta^{-1}A^-cd^*A^-,$$

$$\textit{when } \beta \neq 0 \quad \textit{and either} \quad c \in R(A) \quad \textit{or} \quad d \in R(A^*);$$

$$(A + cd^*)^- = A^- - \frac{A^-cc^*E}{c^*Ec},$$

$$\textit{when } \beta = 0, \quad c \notin R(A), \quad d \in R(A^*);$$

$$(A + cd^*)^- = A^-, \quad \textit{when} \quad \beta = 0, \quad c \in R(A), \quad d \in R(A^*).$$

Proof. Each case may be verified by direct computation.

5. The order of computations. In order to indicate how the results of § 3 may be used in the computation of the Moore–Penrose inverse of a modified matrix, some general comments on computations and the following simple example are provided. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and assume $A^\dagger$ has been previously calculated as

$$A^\dagger = \frac{1}{12}\begin{bmatrix} 3 & -3 & 0 \\ 3 & 5 & 4 \\ 3 & -7 & 4 \\ 3 & -7 & -8 \end{bmatrix}.$$

Suppose -1 is added to the (3.3)-entry of A in order to produce the modified matrix $\tilde{A} = A + cd^*$, where

$$c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad d^* = [0, 0, -1, 0].$$

In general, to compute $\tilde{A}^\dagger$, one must first determine which of the six cases to use. Begin by computing k and h. This is always easy whenever $\tilde{A}$ arises by modifying a single element of A, because if α is added to the (i, j)th entry of A to produce $\tilde{A}$ then k may be taken as the ith column of $A^\dagger$ and h as just α times the jth row of $A^\dagger$. β is then easily computed as

$$\beta = 1 + d^*k \quad \text{or as} \quad \beta = 1 + hc.$$

In our example, $\beta \doteq 2/3$.

In general, the next step is to compute u and v as

$$u = c - Ak \quad \text{and} \quad v = d^* - hA.$$

It is well known that $c \in R(A)$ if and only if $u = 0$, and $d \in R(A^*)$ if and only if $v = 0$. (Nowhere do the matrix products $AA^\dagger$ or $A^\dagger A$ need to be explicitly computed.)

In our example,

$$u = 0 \quad \text{and} \quad v = \tfrac{1}{12}[3, -1, -1, -1],$$

so that Theorem 3 must be used to compute $\tilde{A}^\dagger$.

The terms appearing in Theorem 3 are now easily computed and are as follows:

$$p_1 = -\frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \qquad q_1^* = -\tfrac{1}{8}[-2, 5, -2],$$

$$\frac{1}{\beta}v^*k^*A^\dagger = \frac{1}{24}\begin{bmatrix} 0 & 3 & 6 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

and

$$\tilde{A}^\dagger = (A + cd^*)^\dagger = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 2 & 1 & 2 \\ 2 & -5 & 2 \\ 0 & 0 & -6 \end{bmatrix}.$$

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